

LOG-SOBOLEV INEQUALITIES: DIFFERENT ROLES OF RIC AND HESS

BY FENG-YU WANG¹

Beijing Normal University and Swansea University

Let P_t be the diffusion semigroup generated by $L := \Delta + \nabla V$ on a complete connected Riemannian manifold with $\text{Ric} \geq -(\sigma^2 \rho_o^2 + c)$ for some constants $\sigma, c > 0$ and ρ_o the Riemannian distance to a fixed point. It is shown that P_t is hypercontractive, or the log-Sobolev inequality holds for the associated Dirichlet form, provided $-\text{Hess}_V \geq \delta$ holds outside of a compact set for some constant $\delta > (1 + \sqrt{2})\sigma\sqrt{d-1}$. This indicates, at least in finite dimensions, that Ric and $-\text{Hess}_V$ play quite different roles for the log-Sobolev inequality to hold. The supercontractivity and the ultracontractivity are also studied.

1. Introduction. Let M be a d -dimensional completed connected non-compact Riemannian manifold and $V \in C^2(M)$ such that

$$(1.1) \quad Z := \int_M e^{V(x)} dx < \infty,$$

where dx is the volume measure on M . Let $\mu(dx) = Z^{-1} e^{V(x)} dx$. Under (1.1) it is easy to see that $H_0^{2,1}(\mu) = W^{2,1}(\mu)$, where $H_0^{2,1}(\mu)$ is the completion of $C_0^1(M)$ under the Sobolev norm $\|f\|_{2,1} := \mu(f^2 + |\nabla f|^2)^{1/2}$, and $W^{2,1}(\mu)$ is the completion of the class $\{f \in C^1(M) : f + |\nabla f| \in L^2(\mu)\}$ under $\|\cdot\|_{2,1}$. Then the L -diffusion process is nonexplosive and its semigroup P_t is uniquely determined. Moreover, P_t is symmetric in $L^2(\mu)$ so that μ is P_t -invariant. It is well known by the Bakry–Emery criterion (see [4]) that

$$(1.2) \quad \text{Ric} - \text{Hess}_V \geq K$$

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for some constant $K > 0$ implies the Gross log-Sobolev inequality [14],

$$(1.3) \quad \begin{aligned} \mu(f^2 \log f^2) &:= \int_M f^2 \log f^2 d\mu \leq C\mu(|\nabla f|^2), \\ \mu(f^2) &= 1, f \in C^1(M) \end{aligned}$$

for $C = 2/K$. This result was extended by Chen and the author [9] to the situation that $\text{Ric} - \text{Hess}_V$ is uniformly positive outside a compact set. In the case that $\text{Ric} - \text{Hess}_V$ is bounded below, sufficient concentration conditions of μ for (1.3) to hold are presented in [1, 19, 20]. Obviously, in a condition on $\text{Ric} - \text{Hess}_V$ the Ricci curvature and $-\text{Hess}_V$ play the same role.

What can we do when $\text{Ric} - \text{Hess}_V$ is unbounded below? It seems very hard to confirm the log-Sobolev inequality with the unbounded below condition of $\text{Ric} - \text{Hess}_V$. Therefore, in this paper we try to clarify the roles of Ric and $-\text{Hess}_V$ in the study of the log-Sobolev inequality. Let us first recall the gradient estimate of P_t , which is a key point in the above references to prove the log-Sobolev inequality.

Let x_t be the L -diffusion process starting at x , and let $v \in T_x M$. Due to Bismut [6] and Elworthy–Li [11], under a reasonable lower bound condition of $\text{Ric} - \text{Hess}_V$, one has

$$\langle \nabla P_t f, v \rangle = \mathbb{E} \langle \nabla f(x_t), v_t \rangle, \quad t > 0, f \in C_b^1(M),$$

where $v_t \in T_{x_t} M$ solves the equation

$$D_t v_t := //_{t \rightarrow 0}^{-1} \frac{d}{dt} //_{t \rightarrow 0} v_t = -(\text{Ric} - \text{Hess}_V)^\#(v_t)$$

for $//_{t \rightarrow 0} : T_{x_t} M \rightarrow T_x M$ the associated stochastic parallel displacement, and $(\text{Ric} - \text{Hess}_V)^\#(v_t) \in T_{x_t} M$ with

$$\langle (\text{Ric} - \text{Hess}_V)^\#(v_t), X \rangle := (\text{Ric} - \text{Hess}_V)(v_t, X), \quad X \in T_{x_t} M.$$

Thus, for the gradient of P_t , which is a short distance behavior of the diffusion process, a condition on $\text{Ric} - \text{Hess}_V$ appears naturally.

On the other hand, however, Ric and $-\text{Hess}_V$ play very different roles for long distance behaviors. For instance, Let ρ_o be the Riemannian distance function to a fixed point $o \in M$. If $\text{Ric} \geq -k$ and $-\text{Hess}_V \geq \delta$ for some $k \geq 0$, $\delta \in \mathbb{R}$, the Laplacian comparison theorem implies

$$L\rho_o \leq \sqrt{k(d-1)} \coth[\sqrt{k/(d-1)}\rho_o] - \delta\rho_o.$$

Therefore, for large ρ_o , the Ric lower bound leads to a bounded term while that of $-\text{Hess}_V$ provides a linear term. The same phenomena appears in the formula on distance of coupling by parallel displacement (cf. [3], (2.3), (2.4)), which implies the above Bismut–Elworthy–Li formula by letting the initial distance tend to zero (cf. [15]). Here, $k \geq 0$ is essential for our framework,

since the manifold has to be compact, if Ric is bounded below by a positive constant.

Since the log-Sobolev inequality is always available on bounded regular domains, it is more likely a long-distance property of the diffusion process. So, Ric and $-\text{Hess}_V$ should take different roles in the study of the log-Sobolev inequality. Indeed, it has been observed by the author [20] that (1.3) holds for some $C > 0$, provided Ric is bounded below and $-\text{Hess}_V$ is uniformly positive outside a compact set. This indicates that for the log-Sobolev inequality, the positivity of $-\text{Hess}_V$ is a dominative condition, which allows the Ricci curvature to be bounded below by an arbitrary negative constant, and hence, allows Ric $-\text{Hess}_V$ to be globally negative on M .

The first aim of this paper is to search for the weakest possibility of curvature lower bound for the log-Sobolev inequality to hold under the condition

$$(1.4) \quad -\text{Hess}_V \geq \delta \quad \text{outside a compact set}$$

for some constant $\delta > 0$. This condition is reasonable as the log-Sobolev inequality implies $\mu(e^{\lambda \rho_o^2}) < \infty$ for some $\lambda > 0$ (see, e.g., [2, 17]).

According to the following Theorem 1.1 and Example 1.1, we conclude that under (1.4) the optimal curvature lower bound condition for (1.3) to hold is

$$(1.5) \quad \inf_M \{\text{Ric} + \sigma^2 \rho_o^2\} > -\infty$$

for some constant $\sigma > 0$, such that $\delta > (1 + \sqrt{2})\sigma\sqrt{d-1}$. More precisely, let $\theta_0 > 0$ be the smallest positive constant, such that for any connected complete noncompact Riemannian manifold M and $V \in C^2(M)$, such that $Z := \int_M e^{V(x)} dx < \infty$, the conditions (1.4) and (1.5) with $\delta > \sigma\theta_0\sqrt{d-1}$, implies (1.3) for some $C > 0$. Due to Theorem 1.1 and Example 1.1 below, we conclude that

$$\theta_0 \in [1, 1 + \sqrt{2}].$$

The exact value of θ_0 is however unknown.

THEOREM 1.1. *Assume that (1.4) and (1.5) hold for some constants $c, \delta, \sigma > 0$ with $\delta > (1 + \sqrt{2})\sigma\sqrt{d-1}$. Then (1.3) holds for some $C > 0$.*

EXAMPLE 1.1. Let $M = \mathbb{R}^2$ be equipped with the rotationally symmetric metric

$$ds^2 = dr^2 + \{r e^{kr^2}\}^2 d\theta^2,$$

under the polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{S}^1$ at 0, where $k > 0$ is a constant, then (see, e.g., [13])

$$\text{Ric} = -\frac{(d^2/dr^2)(r e^{kr^2})}{r e^{kr^2}} = -4k - 4k^2 r^2.$$

Thus, (1.5) holds for $\sigma = 2k$. Next, take $V = -k\rho_o^2 - \lambda(\rho_o^2 + 1)^{1/2}$ for some $\lambda > 0$. By the Hessian comparison theorem and the negativity of the sectional curvature, we obtain (1.4) for $\delta = 2k$. Since $d = 2$ and

$$(1.6) \quad e^{V(x)} dx = r e^{-\lambda(1+r^2)^{1/2}} dr d\theta,$$

one has $Z < \infty$ and $\delta = 2k = \sigma\sqrt{d-1}$. But the log-Sobolev inequality is not valid since by Herbst's inequality it implies $\mu(e^{r\rho_o^2}) < \infty$ for some $r > 0$, which is, however, not the case due to (1.6). Since in this example one has $\delta > \sigma\theta\sqrt{d-1}$ for any $\theta < 1$, according to the definition of θ_0 , we conclude that $\theta_0 \geq 1$.

Following the line of [19, 20], the key point in the proof of Theorem 1.1 will be a proper Harnack inequality of type

$$(P_t f(x))^\alpha \leq C_\alpha(t, x, y) P_t f^\alpha(y), \quad t > 0, x, y \in M,$$

for any nonnegative $f \in C_b(M)$, where $\alpha > 1$ is a constant and $C_\alpha \in C((0, \infty), M^2)$ is a positive function. Such an inequality was established in [19] for $\text{Ric} - \text{Hess}_V$ bounded below and extended in [3] to a more general situation with Ric satisfying (1.5).

The Harnack inequality presented in [3] contains a leading term $\exp[\rho(x, y)^4]$, which is, however, too large to be integrability w.r.t. $\mu \times \mu$ under our conditions. So, to prove Theorem 1.1, we shall present a sharper Harnack inequality in Section 3 by refining the coupling method introduced in [3] (see Proposition 3.1 below). This inequality, together with the concentration of μ ensured by (1.4) and (1.5), will imply the hypercontractivity of P_t . To establish this new Harnack inequality, some necessary preparations are presented in Section 2.

Finally, in the same spirit of Theorem 1.1, the supercontractivity and ultracontractivity of P_t are studied in Section 4 under explicit conditions on Ric and $-\text{Hess}_V$.

2. Preparations. We first study the concentration of μ by using (1.4) and (1.5), for which we need to estimate $L\rho_o$ from above according to [5] and references within.

LEMMA 2.1. *If (1.4) and (1.5) hold, then there exists a constant $C_1 > 0$ such that*

$$(2.1) \quad L\rho_o^2 \leq C_1(1 + \rho_o) - 2(\delta - \sigma\sqrt{d-1})\rho_o^2$$

holds outside $\text{cut}(o)$, the cut-locus of o . If moreover $\delta > \sigma\sqrt{d-1}$ then $Z < \infty$ and $\mu(e^{\lambda\rho_o^2}) < \infty$ for all $\lambda < \frac{1}{2}(\delta - \sigma\sqrt{d-1})$.

PROOF. By (1.5) we have $\text{Ric} \geq -(c + \sigma^2 \rho_o^2)$ for some constant $c > 0$. By the Laplacian comparison theorem this implies that

$$\Delta \rho_o \leq \sqrt{(c + \sigma^2 \rho_o^2)(d-1)} \coth[\sqrt{(c + \sigma^2 \rho_o^2)/(d-1)} \rho_o]$$

holds outside $\text{cut}(o)$. Thus, outside $\text{cut}(o)$ one has

$$(2.2) \quad \begin{aligned} \Delta \rho_o^2 &\leq 2\rho_o \sqrt{(c + \sigma^2 \rho_o^2)(d-1)} \coth[\sqrt{(c + \sigma^2 \rho_o^2)/(d-1)} \rho_o] + 2 \\ &\leq 2d + 2\rho_o \sqrt{(c + \sigma^2 \rho_o^2)(d-1)}, \end{aligned}$$

where the second inequality follows from the fact that

$$r \cosh r \leq (1+r) \sinh r, \quad r \geq 0.$$

On the other hand, for $x \notin \text{cut}(o)$ and U the unit tangent vector along the unique minimal geodesic ℓ from o to x , by (1.4) there exists a constant $c_1 > 0$ independent of x such that

$$\langle \nabla V, \nabla \rho_o \rangle(x) = \langle \nabla V, U \rangle(o) + \int_0^{\rho_o(x)} \text{Hess}_V(U, U)(\ell_s) ds \leq c_1 - \delta \rho_o(x).$$

Combining this with (2.2) we prove (2.1).

Finally, let $\delta > \sigma\sqrt{d-1}$ and $0 < \lambda < \frac{1}{2}(\delta - \sigma\sqrt{d-1})$. By (2.1) we have

$$\begin{aligned} L e^{\lambda \rho_o^2} &\leq \lambda e^{\lambda \rho_o^2} (C_1(1 + \rho_o) - 2(\delta - \sigma\sqrt{d-1})\rho_o^2 + 4\lambda \rho_o^2) \\ &\leq c_2 - c_3 \rho_o^2 e^{\lambda \rho_o^2} \end{aligned}$$

for some constants $c_2, c_3 > 0$. By [5], Proposition 3.2, this implies $Z < \infty$ and

$$\int_M \rho_o^2 e^{\lambda \rho_o^2} d\mu \leq \frac{c_2}{c_3} < \infty. \quad \square$$

LEMMA 2.2. *Let x_t be the L -diffusion process with $x_0 = x \in M$. If (1.4) and (1.5) hold with $\delta > \sigma\sqrt{d-1}$, then for any $\delta_0 \in (\sigma\sqrt{d-1}, \delta)$ there exists a constant $C_2 > 0$ such that*

$$\begin{aligned} &\mathbb{E} \exp \left[\frac{(\delta_0 - \sigma\sqrt{d-1})^2}{4} \int_0^T \rho_o(x_t)^2 dt \right] \\ &\leq \exp \left[C_2 T + \frac{1}{4} (\delta_0 - \sigma\sqrt{d-1}) \rho_o(x)^2 \right], \quad T > 0, x \in M. \end{aligned}$$

PROOF. By Lemma 2.1, we have

$$L \rho_o^2 \leq C - 2(\delta_0 - \sigma\sqrt{d-1}) \rho_o^2$$

outside cut(o) for some constant $C > 0$. Then the Itô formula for $\rho_o(x_t)$ due to Kendall [16] implies that

$$(2.3) \quad d\rho_o^2(x_t) \leq 2\sqrt{2}\rho_o(x_t) db_t + [C - 2(\delta_0 - \sigma\sqrt{d-1})\rho_o^2(x_t)] dt$$

holds for some Brownian motion b_t on \mathbb{R} . This implies that the L -diffusion process is nonexplosive so that

$$T_n := \inf\{t \geq 0 : \rho_o(x_t) \geq n\} \rightarrow \infty$$

as $n \rightarrow \infty$. Indeed, (2.3) implies that

$$n\mathbb{P}(T_n \leq t) \leq \mathbb{E}\rho_o(x_{t \wedge T_n})^2 \leq \rho_o(x)^2 + Ct, \quad n \geq 1, t > 0.$$

Hence, $\mathbb{P}(T_n \leq t) \rightarrow 0$ as $n \rightarrow \infty$ for any $t > 0$. This implies $\lim_{n \rightarrow \infty} T_n = \infty$ a.s.

For any $\lambda > 0$ and $n \geq 1$, it follows from (2.3) that

$$\begin{aligned} & \mathbb{E} \exp \left[2\lambda(\delta_0 - \sigma\sqrt{d-1}) \int_0^{T \wedge T_n} \rho_o^2(x_t) dt \right] \\ & \leq e^{\lambda\rho_o^2(x) + C\lambda T} \mathbb{E} \exp \left[2\sqrt{2}\lambda \int_0^{T \wedge T_n} \rho_o(x_t) db_t \right] \\ & \leq e^{\lambda\rho_o^2(x) + C\lambda T} \left(\mathbb{E} \exp \left[16\lambda^2 \int_0^{T \wedge T_n} \rho_o^2(x_t) dt \right] \right)^{1/2}, \end{aligned}$$

where in the last step we have used the inequality

$$\mathbb{E} e^{M_t} \leq (\mathbb{E} e^{2\langle M \rangle_t})^{1/2}$$

for $M_t = 2\sqrt{2}\lambda \int_0^{t \wedge T_n} \rho_o(X_s) db_s$. This follows immediately from the Schwartz inequality and the fact that $\exp[2M_t - 2\langle M \rangle_t]$ is a martingale. Thus, taking

$$\lambda = \frac{1}{8}(\delta_0 - \sigma\sqrt{d-1}),$$

we obtain

$$\begin{aligned} & \mathbb{E} \exp \left[\frac{1}{4}(\delta_0 - \sigma\sqrt{d-1})^2 \int_0^{T \wedge T_n} \rho_o^2(x_t) dt \right] \\ & \leq \exp \left[\frac{1}{4}(\delta_0 - \sigma\sqrt{d-1})\rho_o^2(x) + C_2 T \right] \end{aligned}$$

for some $C_2 > 0$. Then the proof is completed by letting $n \rightarrow \infty$. \square

Finally, we recall the coupling argument introduced in [3] for establishing the Harnack inequality of P_t .

Let $T > 0$ and $x \neq y \in M$ be fixed. Then the L -diffusion process starting from x can be constructed by solving the following Itô stochastic differential equation:

$$d_I x_t = \sqrt{2}\Phi_t dB_t + \nabla V(x_t) dt, \quad x_0 = x,$$

where d_I is the Itô differential on manifolds introduced in [12] (see also [3]), B_t is the d -dimensional Brownian motion, and Φ_t is the horizontal lift of x_t onto the orthonormal frame bundle $O(M)$.

To construct another diffusion process y_t starting from y such that $x_T = y_T$, as in [3], we add an additional drift term to the equation (as explained in [3], Section 3, we may and do assume that the cut-locus of M is empty)

$$d_I y_t = \sqrt{2} P_{x_t, y_t} \Phi_t dB_t + \nabla V(y_t) dt + \xi_t U(x_t, y_t) 1_{\{t < \tau\}} dt, \quad y_0 = y,$$

where P_{x_t, y_t} is the parallel transformation along the unique minimal geodesic ℓ from x_t to y_t , $U(x_t, y_t)$ is the unit tangent vector of ℓ at y_t , $\xi_t \geq 0$ is a smooth function of x_t to be determined, and

$$\tau := \inf\{t \geq 0 : x_t = y_t\}$$

is the coupling time. Since all terms involved in the equation are regular enough, there exists a unique solution y_t . Furthermore, since the additional term containing $1_{\{t < \tau\}}$ vanishes from the coupling time on, one has $x_t = y_t$ for $t \geq \tau$ due to the uniqueness of solutions.

LEMMA 2.3. *Assume that (1.4) and (1.5) hold with $\delta \geq 2\sigma\sqrt{d-1}$. Then there exists a constant $C_3 > 0$ independent of x, y and T such that $x_T = y_T$ holds for $\xi_t := C_3 + 2\sigma\sqrt{d-1}\rho_o(x_t) + \frac{\rho(x, y)}{T}$.*

PROOF. According to Section 2 in [3], we have

$$(2.4) \quad \begin{aligned} d\rho(x_t, y_t) = & \{I(x_t, y_t) + \langle \nabla V, \nabla \rho(\cdot, y_t) \rangle(x_t) \\ & + \langle \nabla V, \nabla \rho(x_t, \cdot) \rangle(y_t) - \xi_t\} dt, \quad t < \tau, \end{aligned}$$

where

$$I_Z(x_t, y_t) = \sum_{i=1}^{d-1} \int_0^{\rho(x_t, y_t)} (|\nabla_U J_i|^2 - \langle R(U, J_i)U, J_i \rangle)(\ell_s) ds$$

for R the Riemann curvature tensor, U the unit tangent vector of the minimal geodesic $\ell: [0, \rho(x_t, y_t)] \rightarrow M$ from x_t to y_t , and $\{J_i\}_{i=1}^{d-1}$ the Jacobi fields along ℓ , which, together with U , consist of an orthonormal basis of the tangent space at x_t and y_t and satisfy

$$J_i(y_t) = P_{x_t, y_t} J_i(x_t), \quad i = 1, \dots, d-1.$$

By (1.5) we take a constant $c \geq 0$ such that $\text{Ric} \geq -(c + \sigma^2 \rho_o^2)$. Letting

$$K(x_t, y_t) = \sup_{\ell([0, \rho(x_t, y_t)])} \{c + \sigma^2 \rho_o^2\},$$

we obtain from Wang [21], Theorem 2.14 (see also [7, 8]), that

$$(2.5) \quad I(x_t, y_t) \leq 2\sqrt{K(x_t, y_t)(d-1)} \tanh\left[\frac{\rho(x_t, y_t)}{2}\sqrt{K(x_t, y_t)/(d-1)}\right].$$

Moreover, by (1.4) there exist two constants $r_0, r_1 > 0$ such that $-\text{Hess}_V \geq \delta$ outside $B(o, r_0)$ but $\leq r_1$ on $B(o, r_0)$, where $B(o, r_0)$ is the closed geodesic ball at o with radius r_0 . Since the length of ℓ contained in $B(o, r_0)$ is less than $2r_0$, we conclude that

$$\begin{aligned} & \langle \nabla V, \nabla \rho(\cdot, y_t) \rangle(x_t) + \langle \nabla V, \nabla \rho(x_t, \cdot) \rangle(y_t) \\ &= \int_0^{\rho(x_t, y_t)} \text{Hess}_V(U, U)(\ell_s) ds \leq 2r_0 r_1 - (\rho(x_t, y_t) - 2r_0)^+ \delta \\ &\leq c_1 - \delta \rho(x_t, y_t) \end{aligned}$$

for some constant $c_1 > 0$. Combining this with (2.4), (2.5) and

$$\xi_t = C_3 + 2\sigma\sqrt{d-1} \rho_o(x_t) + \frac{\rho(x, y)}{T},$$

we arrive at

$$\begin{aligned} d\rho(x_t, y_t) &\leq \left\{ 2\sqrt{K(x_t, y_t)(d-1)} + c_1 - \delta \rho(x_t, y_t) \right. \\ &\quad \left. - C_3 - 2\sigma\sqrt{d-1} \rho_o(x_t) - \frac{\rho(x, y)}{T} \right\} dt \end{aligned}$$

for $t < \tau$. Noting that

$$\begin{aligned} \sqrt{K(x_t, y_t)} &\leq (c + \sigma^2[\rho_o(x_t) + \rho(x_t, y_t)]^2)^{1/2} \\ &\leq \sqrt{c} + \sigma[\rho_o(x_t) + \rho(x_t, y_t)], \end{aligned}$$

and $\delta \geq 2\sigma\sqrt{d-1}$, one has

$$2\sqrt{K(x_t, y_t)(d-1)} - \delta \rho(x_t, y_t) - 2\sigma\sqrt{d-1} \rho_o(x_t) \leq 2\sqrt{c(d-1)}.$$

Thus, when $C_3 \geq c_1 + 2\sqrt{c(d-1)}$ we have

$$d\rho(x_t, y_t) \leq -\frac{\rho(x, y)}{T} dt, \quad t < \tau,$$

so that

$$0 = \rho(x_\tau, y_\tau) \leq \rho(x, y) - \int_0^\tau \frac{\rho(x, y)}{T} dt = \frac{T-\tau}{T} \rho(x, y),$$

which implies that $\tau \leq T$ and hence, $x_T = y_T$. \square

3. Harnack inequality and proof of Theorem 1.1. We first prove the following Harnack inequality using results in Section 2.

PROPOSITION 3.1. Assume that (1.4) and (1.5) hold with $\delta > (1 + \sqrt{2})\sigma \times \sqrt{d-1}$. Then there exist $C > 0$ and $\alpha > 1$ such that

$$(3.1) \quad (P_T f(y))^\alpha \leq (P_T f^\alpha(x)) \exp \left[\frac{C}{T} \rho(x, y)^2 + C(T + \rho_o(x)^2) \right]$$

holds for all $x, y \in M, T > 0$ and nonnegative $f \in C_b(M)$.

PROOF. According to Lemma 2.3, we take

$$\xi_t = C_3 + 2\sigma\sqrt{d-1}\rho_o(x_t) + \frac{\rho(x, y)}{T},$$

such that $\tau \leq T$ and $x_T = y_T$. Obviously, y_t solves the equation

$$d_I y_t = \sqrt{2}\tilde{\Phi}_t d\tilde{B}_t + \nabla V(y_t) dt$$

for $\tilde{\Phi}_t := P_{x_t, y_t} \Phi_t$ being the horizontal lift of y_t , and \tilde{B}_t solving the equation

$$d\tilde{B}_t = dB_t + \frac{1}{\sqrt{2}} \tilde{\Phi}_t^{-1} \xi_t U(x_t, y_t) 1_{\{t < \tau\}} dt.$$

By the Girsanov theorem and the fact that $\tau \leq T$, the process $\{\tilde{B}_t : t \in [0, T]\}$ is a d -dimensional Brownian motion under the probability measure $R\mathbb{P}$ for

$$R := \exp \left[-\frac{1}{\sqrt{2}} \int_0^\tau \langle P_{x_t, y_t} \Phi_t dB_t, \xi_t U(x_t, y_t) \rangle - \frac{1}{4} \int_0^\tau \xi_t^2 dt \right].$$

Thus, under this probability measure $\{y_t : t \in [0, T]\}$ is generated by L . In particular, $P_T f(y) = \mathbb{E}[f(y_T)R]$. Combining this with the Hölder inequality and noting that $x_T = y_T$, we obtain

$$\begin{aligned} P_T f(y) &= \mathbb{E}[f(y_T)R] = \mathbb{E}[f(x_T)R] \\ &\leq (P_T f^\alpha(x))^{1/\alpha} (\mathbb{E} R^{\alpha/(\alpha-1)})^{(\alpha-1)/\alpha}. \end{aligned}$$

That is,

$$(3.2) \quad (P_T f(y))^\alpha \leq (P_T f^\alpha(x)) (\mathbb{E} R^{\alpha/(\alpha-1)})^{\alpha-1}.$$

Since for any continuous exponential integrable martingale M_t and any $\beta, p > 1$, the process $\exp[\beta p M_t - \frac{p^2 \beta^2}{2} \langle M \rangle_t]$ is a martingale, by the Hölder inequality one has

$$\begin{aligned} (3.3) \quad \mathbb{E} e^{\beta M_t - (\beta/2) \langle M \rangle_t} &= \mathbb{E} [e^{\beta M_t - (\beta^2 p/2) \langle M \rangle_t} \cdot e^{(\beta(\beta p - 1)/2) \langle M \rangle_t}] \\ &\leq \mathbb{E} (e^{(\beta p(\beta p - 1)/(2(p-1))) \langle M \rangle_t})^{(p-1)/p}. \end{aligned}$$

By taking $\beta = \alpha/(\alpha - 1)$ we obtain

$$\begin{aligned} (3.4) \quad &(\mathbb{E} R^{\alpha/(\alpha-1)})^{\alpha-1} \\ &\leq \left\{ \mathbb{E} \exp \left[\frac{p\alpha(p\alpha - \alpha + 1)}{8(p-1)(\alpha-1)^2} \int_0^T \xi_t^2 dt \right] \right\}^{(\alpha-1)(p-1)/p}, \quad p > 1. \end{aligned}$$

Since $\delta > (1 + \sqrt{2})\sigma\sqrt{d-1}$, we may take $\delta_0 \in ((1 + \sqrt{2})\sigma\sqrt{d-1}, \delta)$, small $\varepsilon' > 0$ and large $C_4 > 0$, independent of T, x and y , such that

$$\begin{aligned}\xi_t^2 &= \left(C_3 + 2\sigma\sqrt{d-1}\rho_o(x_t) + \frac{\rho(x, y)}{T} \right)^2 \\ &\leq (1 - \varepsilon') \left[C_4 + \frac{C_4\rho(x, y)^2}{T^2} + 2(\delta_0 - \sigma\sqrt{d-1})^2\rho_o(x_t)^2 \right]\end{aligned}$$

holds. Moreover, since

$$(3.5) \quad \lim_{p \downarrow 1} \lim_{\alpha \uparrow \infty} \frac{p\alpha(p\alpha - \alpha + 1)}{8(p-1)(\alpha-1)^2} = \frac{1}{8},$$

there exist $p, \alpha > 1$ such that

$$\begin{aligned}\frac{p\alpha(p\alpha - \alpha + 1)}{8(p-1)(\alpha-1)^2} \int_0^T \xi_t^2 dt \\ \leq C_4 T + \frac{C_4\rho(x, y)^2}{T} + \frac{(\delta_0 - \sigma\sqrt{d-1})^2}{4} \int_0^T \rho_o(x_t)^2 dt.\end{aligned}$$

Combining this with (3.4) and Lemma 2.2, we obtain

$$(\mathbb{E} R^{\alpha/(\alpha-1)})^{\alpha-1} \leq \exp \left[C_5 T + \frac{C_5\rho(x, y)}{T} + C_5\rho_o(x)^2 \right], \quad T > 0, x \in M,$$

for some constant $C_5 > 0$. This completes the proof by (3.2). \square

PROOF OF THEOREM 1.1. By Proposition 3.1, let $\alpha > 1$ and $C > 0$ such that (3.1) holds. Since $\delta > \sigma\sqrt{d-1}$, we may take $T > 0$ such that

$$\frac{C}{T} \leq \varepsilon := \frac{1}{8}(\delta - \sigma\sqrt{d-1}).$$

Then for any nonnegative $f \in C_b(M)$ with $\mu(f^\alpha) = 1$, since μ is P_T -invariant, it follows from (3.1) that

$$\begin{aligned}1 &= \int_M P_T f^\alpha(x) \mu(dx) \geq (P_T f(y))^\alpha \int_M e^{-\varepsilon\rho(x, y)^2 - C(1+\rho_o(x)^2)} \mu(dx) \\ &\geq (P_T f(y))^\alpha \int_{\{\rho_o \leq 1\}} e^{-\varepsilon(1+\rho_o(y))^2 - 2C} \mu(dx) \\ &\geq \varepsilon' (P_T f(y))^\alpha \exp[-2\varepsilon\rho_o(y)^2], \quad y \in M,\end{aligned}$$

for some constant $\varepsilon' > 0$. Thus,

$$\int_M (P_T f(y))^{2\alpha} \mu(dy) \leq \frac{1}{\varepsilon'} \int_M e^{4\varepsilon\rho_o(y)^2} \mu(dy) < \infty,$$

according to Lemma 2.1. This implies that

$$\|P_T\|_{L^\alpha(\mu) \rightarrow L^{2\alpha}(\mu)} < \infty.$$

Therefore, the log-Sobolev inequality (1.3) holds for some constant $C > 0$, due to the uniformly positively improving property of P_t (see [20], proof of Theorem 1.1, and [1]). \square

4. Supercontractivity and ultracontractivity. Recall that P_t is called supercontractive if $\|P_t\|_{2 \rightarrow 4} < \infty$ for all $t > 0$ while ultracontractive if $\|P_t\|_{2 \rightarrow \infty} < \infty$ for all $t > 0$ (see [10]). In the present framework these two properties are stronger than the hypercontractivity: $\|P_t\|_{2 \rightarrow 4} \leq 1$ for some $t > 0$, which is equivalent to (1.3) due to Gross [14].

PROPOSITION 4.1. *Under (1.4) and (1.5), P_t is supercontractive if and only if $\mu(\exp[\lambda \rho_o^2]) < \infty$ for all $\lambda > 0$, while it is ultracontractive if and only if $\|P_t \exp[\lambda \rho_o^2]\|_\infty < \infty$ for all $t, \lambda > 0$.*

PROOF. The proof is similar to that of [18], Theorem 2.3. Let $f \in L^2(\mu)$ with $\mu(f^2) = 1$. By (3.1) for $\alpha = 2$ and noting that μ is P_t -invariant, we obtain

$$\begin{aligned} 1 &\geq (P_T f(y))^2 \int_M \exp\left[-\frac{C}{T} \rho(x, y)^2 - C(T + \rho_o(x)^2)\right] \mu(dx) \\ &\geq (P_T f(y))^2 \exp\left[-\frac{2C}{T} (\rho_o(y)^2 + 1) - C(T + 1)\right] \mu(B(o, 1)). \end{aligned}$$

Hence, for any $T > 0$ there exists a constant $\lambda_T > 0$ such that

$$(4.1) \quad |P_T f| \leq \exp[\lambda_T(1 + \rho_o^2)], \quad T > 0, \mu(f^2) = 1.$$

(1) If $\mu(e^{\lambda \rho_o^2}) < \infty$ for any $\lambda > 0$, (4.1) yields that

$$\|P_T\|_{2 \rightarrow 4}^4 \leq \mu(e^{4\lambda_T(1 + \rho_o^2)}) < \infty, \quad T > 0.$$

Conversely, if P_t is supercontractive then the super log-Sobolev inequality (cf. [10])

$$\mu(f^2 \log f^2) \leq r \mu(|\nabla f|^2) + \beta(r), \quad r > 0, \mu(f^2) = 1,$$

holds for some $\beta: (0, \infty) \rightarrow (0, \infty)$. By [2] (see also [17, 18]), this inequality implies $\mu(e^{\lambda \rho_o^2}) < \infty$ for all $\lambda > 0$.

(2) By (4.1) and the semigroup property,

$$\|P_T\|_{2 \rightarrow \infty} \leq \|P_{T/2} e^{\lambda_{T/2}(1 + \rho_o^2)}\|_\infty < \infty, \quad T > 0,$$

provided $\|P_t e^{\lambda \rho_o^2}\|_\infty < \infty$ for any $t, \lambda > 0$. Conversely, since the ultracontractivity is stronger than the supercontractivity, it implies that $e^{\lambda \rho_o^2} \in L^2(\mu)$ for any $\lambda > 0$ as explained above. Therefore,

$$\|P_t e^{\lambda \rho_o^2}\|_\infty \leq \|P_t\|_{2 \rightarrow \infty} \|e^{\lambda \rho_o^2}\|_2 < \infty, \quad \lambda > 0.$$

Then the proof is completed. \square

To derive explicit conditions for the supercontractivity and ultracontractivity, we consider the following stronger version of (1.4):

$$(4.2) \quad -\text{Hess}_V \geq \Phi \circ \rho_o \quad \text{holds outside a compact subset of } M$$

for a positive increasing function Φ with $\Phi(r) \uparrow \infty$ as $r \uparrow \infty$. We then aim to search for reasonable conditions on positive increasing function Ψ such that

$$(4.3) \quad \text{Ric} \geq -\Psi \circ \rho_o$$

implies the supercontractivity and/or ultracontractivity.

THEOREM 4.2. *If (4.3) and (4.2) hold for some increasing positive functions Φ and Ψ such that*

$$(4.4) \quad \lim_{r \rightarrow \infty} \Phi(r) = \lim_{r \rightarrow \infty} \frac{(\int_0^r \Phi(s) ds)^2}{\Phi(r)} = \infty,$$

$$(4.5) \quad \begin{aligned} & \sqrt{\Psi(r+t)(d-1)} \\ & \leq \theta \int_0^r \Phi(s) ds + \frac{1}{2} \int_0^{t/2} \Phi(s) ds + C, \quad r, t \geq 0, \end{aligned}$$

for some constants $\theta \in (0, 1/(1 + \sqrt{2}))$ and $C > 0$. Then P_t is supercontractive. Furthermore, if

$$(4.6) \quad \int_1^\infty \frac{ds}{\sqrt{s} \int_0^{\sqrt{r}} \Phi(u) du} < \infty,$$

then P_t is ultracontractive. More precisely, for

$$\Gamma_1(r) := \frac{1}{\sqrt{r}} \int_0^{\sqrt{r}} \Phi(s) ds, \quad \Gamma_2(r) := \int_r^\infty \frac{ds}{\sqrt{s} \int_0^{\sqrt{s}} \Phi(u) du}, \quad r > 0,$$

(4.6) implies

$$(4.7) \quad \|P_t\|_{2 \rightarrow \infty} \leq \exp \left[c + \frac{c}{t} (1 + \Gamma_1^{-1}(c/t) + \Gamma_2^{-1}(t/c)) \right] < \infty, \quad t > 0,$$

for some constant $c > 0$ and

$$\Gamma_1^{-1}(s) := \inf \{ t \geq 0 : \Gamma_1(t) \geq s \}, \quad s \geq 0.$$

PROOF. (a) Replacing $c + \rho_o^2$ by $\Psi \circ \rho_o$ and noting that $\text{Hess}_V \leq -\Phi \circ \rho_o$ for large ρ_o , the proof of Lemma 2.1 implies

$$(4.8) \quad L\rho_o^2 \leq c_1(1 + \rho_o) - 2\rho_o \left(\int_0^{\rho_o} \Phi(s) ds - \sqrt{\Psi \circ \rho_o(d-1)} \right)$$

for some constant $c_1 > 0$. Combining this with (4.5) and noting that $\frac{1}{\rho_o} \times \int_0^{\rho_o} \Phi(s) ds \rightarrow \infty$ as $\rho_o \rightarrow \infty$, we conclude that for any $\lambda > 0$,

$$(4.9) \quad \begin{aligned} L e^{\lambda \rho_o^2} &\leq C - \frac{2\lambda \rho_o \sqrt{2}}{1 + \sqrt{2}} e^{\lambda \rho_o^2} \int_0^{\rho_o} \Phi(s) ds + 4\lambda^2 \rho_o^2 e^{\lambda \rho_o^2} \\ &\leq C + C(\lambda) - \lambda \rho_o e^{\lambda \rho_o^2} \int_0^{\rho_o} \Phi(s) ds, \end{aligned}$$

where $C > 0$ is a universal constant and

$$(4.10) \quad \begin{aligned} C(\lambda) &:= \sup_{r>0} r e^{\lambda r^2} \left\{ 4\lambda^2 r - \frac{\lambda}{(1 + \sqrt{2})^2} \int_0^r \Phi(s) ds \right\} \\ &= \sup_{r^2 \leq \Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda)} r e^{\lambda r^2} \left\{ 4\lambda^2 r - \frac{\lambda}{(1 + \sqrt{2})^2} \int_0^r \Phi(s) ds \right\} \\ &\leq 4\lambda^2 \Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda) \exp[\lambda \Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda)] \\ &\leq \exp[4\lambda + 2\lambda \Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda)] < \infty. \end{aligned}$$

Therefore, (1.1) holds and

$$(4.11) \quad \mu(e^{\lambda \rho_o^2}) < \infty, \quad \lambda > 0.$$

(b) By (4.5), (4.8) and Kendall's Itô formula [16] as in the proof of Lemma 2.2, we have

$$d\rho_o^2(x_t) \leq 2\sqrt{2}\rho_o(x_t) db_t + \left(C_1 - \frac{2\sqrt{2}\rho_o(x_t)(1 + \varepsilon)}{1 + \sqrt{2}} \int_0^{\rho_o(x_t)} \Phi(s) ds \right) dt$$

for some constants $\varepsilon, C_1 > 0$, where x_t and b_t are in the proof of Lemma 2.2. Let

$$(4.12) \quad \varphi(r) = \int_0^r \frac{ds}{\sqrt{s}} \int_0^{\sqrt{s}} \Phi(u) du, \quad r \geq 0.$$

We arrive at

$$\begin{aligned} d\varphi \circ \rho_o^2(x_t) &\leq 2\sqrt{2}\rho_o(x_t) \varphi' \circ \rho_o^2(x_t) db_t + 4\rho_o^2(x_t) \varphi'' \circ \rho_o^2(x_t) dt \\ &\quad + \varphi' \circ \rho_o^2(x_t) \left(C_1 - \frac{2\sqrt{2}\rho_o(x_t)(1 + \varepsilon)}{1 + \sqrt{2}} \int_0^{\rho_o(x_t)} \Phi(s) ds \right) dt. \end{aligned}$$

From (4.4) we see that

$$\frac{\rho_o \varphi'' \circ \rho_o^2}{\varphi' \circ \rho_o^2 \int_0^{\rho_o} \Phi(s) ds} \leq \frac{\Phi \circ \rho_o}{2(\int_0^{\rho_o} \Phi(s) ds)^2},$$

which goes to zero as $\rho_o \rightarrow \infty$. Then there exists a constant $C_2 > C_1$ such that

$$\begin{aligned} d\varphi \circ \rho_o^2(x_t) &\leq 2\sqrt{2} \left(\int_0^{\rho_o(x_t)} \Phi(s) ds \right) db_t \\ &\quad + C_2 dt - \frac{2\sqrt{2}}{1 + \sqrt{2}} \left(\int_0^{\rho_o(x_t)} \Phi(s) ds \right)^2 dt. \end{aligned}$$

This implies that for any $\lambda > 0$,

$$\begin{aligned} &\mathbb{E} \exp \left[\frac{2\sqrt{2}\lambda}{1 + \sqrt{2}} \int_0^T \left(\int_0^{\rho_o(x_t)} \Phi(s) ds \right)^2 dt \right] \\ &\leq e^{C_2 \lambda T + \lambda \varphi \circ \rho_o^2(x)} \mathbb{E} \exp \left[2\sqrt{2}\lambda \int_0^T \left(\int_0^{\rho_o(x_t)} \Phi(s) ds \right) db_t \right] \\ &\leq e^{C_2 \lambda T + \lambda \varphi \circ \rho_o^2(x)} \left(\mathbb{E} \exp \left[16\lambda^2 \int_0^T \left(\int_0^{\rho_o(x_t)} \Phi(s) ds \right)^2 dt \right] \right)^{1/2}. \end{aligned}$$

Taking

$$\lambda = \frac{\sqrt{2}}{8(1 + \sqrt{2})},$$

we arrive at

$$\begin{aligned} &\mathbb{E} \exp \left[\frac{1}{2(1 + \sqrt{2})^2} \int_0^T \left(\int_0^{\rho_o(x_t)} \Phi(s) ds \right)^2 dt \right] \\ (4.13) \quad &\leq e^{2C_2 T + \varphi \circ \rho_o^2(x) \sqrt{2}/8(1 + \sqrt{2})}. \end{aligned}$$

(c) Let $\gamma: [0, \rho(x_t, y_t)] \rightarrow M$ be the minimal geodesic from x_t to y_t , and U its tangent unit vector. By (4.2), there exists a constant $C_3 > 0$ such that

$$\begin{aligned} &\langle \nabla V, \nabla \rho(\cdot, y_t) \rangle(x_t) + \langle \nabla V, \nabla \rho(x_t, \cdot) \rangle(y_t) \\ (4.14) \quad &= \int_0^{\rho(x_t, y_t)} \text{Hess}_V(U_s, U_s) ds \leq C_3 - \int_0^{\rho(x_t, y_t)/2} \Phi(s) ds. \end{aligned}$$

To understand the last inequality, we assume, for instance, that $\rho_o(x_t) \geq \rho_o(y_t)$ so that by the triangle inequality,

$$\rho_o(\gamma_s) \geq \rho_o(x_t) - s \geq \rho(x_t, y_t)/2 - s, \quad s \in [0, \rho(x_t, y_t)/2].$$

For the coupling constructed in Section 3, one concludes from (4.14) and the proof of Lemma 2.3 that

$$(4.15) \quad d\rho(x_t, y_t) \leq \left\{ 2\sqrt{K(x_t, y_t)(d-1)} + C_4 - \int_0^{\rho(x_t, y_t)/2} \Phi(s) ds - \xi_t \right\} dt, \quad t < \tau,$$

holds for some constant $C_4 > 0$, where

$$K(x_t, y_t) := \sup_{\ell([0, \rho(x_t, y_t)])} \Psi \circ \rho_o \leq \Psi(\rho_o(x_t) + \rho(x_t, y_t)),$$

and ℓ is the minimal geodesic from x_t to y_t . Combining (4.5) and (4.15), we obtain

$$d\rho(x_t, y_t) \leq \left\{ C_4 + 2\theta \int_0^{\rho_o(x_t)} \Phi(s) ds - \xi_t \right\} dt, \quad t < \tau.$$

So, taking

$$\xi_t = C_4 + 2\theta \int_0^{\rho_o(x_t)} \Phi(s) ds + \frac{\rho(x, y)}{T},$$

we arrive at

$$d\rho(x_t, y_t) \leq -\frac{\rho(x, y)}{T} dt, \quad t < \tau.$$

This implies $\tau \leq T$, and hence $x_T = y_T$ a.s.

Combining (4.5) with (3.4) and (3.5) we conclude that for the present choice of ξ_t there exist $\alpha, p, C_5 > 1$ such that

$$\begin{aligned} (\mathbb{E} R^{\alpha/(\alpha-1)})^{p/(p-1)} &\leq \mathbb{E} \exp \left[\frac{1}{2(1+\sqrt{2})^2} \int_0^T \left(\int_0^{\rho(x_t)} \Phi(s) ds \right)^2 dt \right. \\ &\quad \left. + C_5 T + \frac{C_5}{T} \rho(x, y)^2 \right]. \end{aligned}$$

Combining this with (4.13) and (3.2) we obtain

$$(4.16) \quad (P_T f(y))^\alpha \leq (P_T f^\alpha(x)) \exp \left[CT + \frac{C}{T} \rho(x, y)^2 + C\varphi \circ \rho^2(x) \right]$$

holds for some $\alpha, C > 1$, any positive $f \in C_b(M)$ and all $x, y \in M, T > 0$.

(d) For any positive $f \in C_b(M)$ with $\mu(f^\alpha) = 1$, (4.16) implies that

$$(P_T f(y))^\alpha \int_{B(o,1)} \exp \left[-CT - \frac{C}{T} \rho(x, y)^2 - C\varphi^2(x) \right] \mu(dx) \leq 1.$$

Therefore, there exists a constant $C' > 0$ such that

$$(4.17) \quad (P_T f(y))^\alpha \leq \exp \left[C'(1+T) + \frac{C'}{T} \rho(y)^2 \right], \quad y \in M, T > 0.$$

Combining this with (4.11) we obtain

$$\|P_T\|_{\alpha \rightarrow p\alpha} < \infty, \quad T > 0, p > 1.$$

This is equivalent to the supercontractivity by the Riesz–Thorin interpolation theorem and $\|P_t\|_{1 \rightarrow 1} = 1$. Thus, the first assertion holds.

(e) To prove (4.7), it suffices to consider $t \in (0, 1]$ since $\|P_t\|_{2 \rightarrow \infty}$ is decreasing in $t > 0$. So, below we assume that $T \leq 1$. By (4.17) and the fact that $(P_{2T}f)^\alpha \leq P_T(P_T f)^\alpha$, we have

$$(4.18) \quad \|P_{2T}\|_{\alpha \rightarrow \infty} \leq \|P_T e^{2C'\rho_o^2/T}\|_{\infty} e^{C'(1+T)}, \quad T > 0.$$

Therefore, by the Riesz–Thorin interpolation theorem and $\|P_t\|_{1 \rightarrow 1} = 1$, for the ultracontractivity it suffices to show that

$$(4.19) \quad \|P_T e^{\lambda \rho_o^2}\|_{\infty} < \infty, \quad \lambda, T > 0.$$

Since Φ is increasing, it is easy to check that

$$\eta(r) := \sqrt{r} \int_0^{\sqrt{r}} \Phi(s) ds, \quad r \geq 0,$$

is convex, and so is $s \mapsto s\eta(\frac{\log s}{\lambda})$ for $\lambda > 0$. Thus, it follows from (4.9) and the Jensen inequality that

$$h_{\lambda,x}(t) := \mathbb{E} e^{\lambda \rho_o^2(x_t)} < \infty, \quad x_0 = x \in M, \lambda, t > 0,$$

and

$$\frac{d^+}{dt} h_{\lambda,x}(t) \leq C + C(\lambda) - \lambda h_{\lambda,x}(t) \eta(\lambda^{-1} \log h_{\lambda,x}(t)), \quad t > 0.$$

This implies (4.19), provided (4.6) holds. This can be done by considering the following two situations:

(1) Since $h_{\lambda,x}(t)$ is decreasing provided $\lambda h_{\lambda,x}(t) \eta(\lambda^{-1} \log h_{\lambda,x}(t)) > C + C(\lambda)$, if

$$\lambda h_{\lambda,x}(0) \eta(\lambda^{-1} \log h_{\lambda,x}(0)) \leq 2C + 2C(\lambda),$$

then

$$h_{\lambda,x}(t) \leq \sup\{r \geq 1 : \lambda r \eta(\lambda^{-1} \log r) \leq 2C + 2C(\lambda)\} \leq \frac{1}{\lambda} (2C + 2C(\lambda)) + C''$$

for some constant $C'' > 0$.

(2) If $\lambda h_{\lambda,x}(0)\eta(\lambda^{-1}\log h_{\lambda,x}(0)) > 2C + 2C(\lambda)$, then $h_{\lambda,x}(t)$ is decreasing in t up to

$$t_\lambda := \inf\{t \geq 0 : \lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)) \leq 2C + 2C(\lambda)\}.$$

Indeed,

$$\frac{d^+}{dt} h_{\lambda,x}(t) \leq -\frac{\lambda}{2} h_{\lambda,x}(t) \eta(\lambda^{-1}\log h_{\lambda,x}(t)), \quad t \leq t_\lambda.$$

Thus,

$$\int_{h_{\lambda,x}(T \wedge t_\lambda)}^\infty \frac{dr}{r\eta(\lambda^{-1}\log r)} \geq \frac{\lambda}{2}(T \wedge t_\lambda).$$

This is equivalent to

$$\Gamma_2(\lambda^{-1}\log h_{\lambda,x}(T \wedge t_\lambda)) \geq \frac{1}{2}(T \wedge t_\lambda).$$

Hence,

$$h_{\lambda,x}(T \wedge t_\lambda) \leq \exp[\lambda\Gamma_2^{-1}(\frac{1}{2}(T \wedge t_\lambda))].$$

Since it is reduced to case (1) if $T > t_\lambda$ by regarding t_λ as the initial time, in conclusion we have

$$\sup_{x \in M} h_{\lambda,x}(T) \leq \max\left\{\exp[\lambda\Gamma_2^{-1}(T/2)], C'' + \frac{1}{\lambda}(2C + 2C(\lambda))\right\}.$$

Therefore, (4.7) follows from (4.18), (4.10) with $\lambda = 2C'/T$, and the Riesz interpolation theorem. \square

Finally, we note that a simple example for conditions in Theorem 4.2 to hold is

$$\Phi(s) = s^{\alpha-1}, \quad \Psi(s) = \varepsilon s^{2\alpha}$$

for $\alpha > 1$ and small enough $\varepsilon > 0$. In this case P_t is ultracontractive with

$$\|P_t\|_{2 \rightarrow \infty} \leq \exp[c(1 + t^{-(\alpha+1)/(\alpha-1)})], \quad t > 0,$$

for some $c > 0$.

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SCHOOL OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING 100875
CHINA
AND
SWANSEA UNIVERSITY
SINGLETON PARK
SWANSEA, SA2 8PP
UNITED KINGDOM
E-MAIL: wangfy@bnu.edu.cn